$$\Phi_2(z) = \frac{D}{48}\left(14a\cos\delta + \frac{D}{2}\right)\sin 2(kz + \alpha_0),$$

$$\Phi_3(z) = \frac{E}{8w}\left[2(1 - a\cos\delta)F_2(y^*) - 3\ln f^* - \frac{3}{w_1}F_3(y^*)\right]\sin 2(kz + \alpha_0) +$$

$$+ \frac{D}{8}y^*\left[2a\cos\delta\cdot G_1(y^*) - DG_2(y^*)\right]\sin 2(kz + \alpha_0),$$

$$\Phi_4(z) = -\frac{D}{240}\left(47a\cos\delta + \frac{7D}{3}\right)\cos 2(kz + \alpha_0),$$

$$\Phi_5(z) = -\frac{E}{4wy^*}\left[3J_4(y^*) + 2(1 - a\cos\delta)J_2(y^*) - \frac{3}{w_1}J_3(y^*)\right]\cos 2(kz + \alpha_0) -$$

$$-\frac{Dy^{*2}}{4}\left[2a\cos\delta\cdot I_1(y^*) - DI_2(y^*)\right]\cos 2(kz + \alpha_0).$$

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RADIATION OF INTERNAL GRAVITATIONAL WAVES IN THE CASE OF UNIFORM MOTION OF SOURCES OF VARIABLE AMPLITUDE (THE PLANE PROBLEM)

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A uniformly moving source generates waves similar to Cherenkov radiation. In a liquid with stratified density these are internal gravitational waves. A fixed oscillating source generates another type of radiation of gravitational waves. When a source of variable amplitude is moving the variety of excited waves increases. Wave-forerunners appear which carry away energy in the direction of motion with a velocity exceeding the velocity of the source.

The linear wave fields around an oscillating moving source were analyzed in [1-7] for the simplest types of stratification, a free surface and a discontinuous jump in the density. Below we estimate the energy losses of such sources for a more general form of stratification. The method of energy estimates also enables one to investigate more simply the main known and certain additional features of the radiation in the case of discontinuous stratification.

In considering a mass source with an harmonically varying amplitude, moving uniformly horizontally in a stratified incompressible liquid, we will confine ourselves to analyzing the

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plane problem. Beginning with the case of discontinuous stratification (a free surface, in particular) of infinite and finite depth, we will then consider the radiation of waves in a waveguide of finite depth with an arbitrary stratification and in an unbounded uniformly stratified liquid. The number of excited modes increases rapidly as the stratification becomes more complicated.

1. The Potential Flow of a Piecewise-Uniform Liquid. The velocity potential φ is found by solving Poisson's equation with a moving oscillating mass source

$$\Delta\varphi = \nabla\cdot v = m(t, x, z) = m_0(x - v_0 t, z)\exp(-i\omega_0 t),$$

supplemented by the boundary conditions that the perturbations of the velocity $v = \nabla\varphi$ should fall off far from the source, that the vertical component of the velocity on the horizontal solid bottom should be zero, and the pressure and vertical displacement on the surface of a jump in density should be continuous.

For an arbitrary distribution of the sources, the solution can be represented by an integral convolution of the sources with a Fourier transformation with respect to the delay of Green's function $G(t, x; z, z')$

$$\varphi(t, x, z) = \exp(-i\omega_0 t)\int dx'\,dz'\,g(x - v_0 t - x'; z, z')\,m_0(x', z'),$$

$$g(x; z, z') = \int dt'\,G(t', x + v_0 t'; z, z')\exp(i\omega_0 t'),$$

which is a solution of the same problem for an instantaneous point source with the delay condition

$$G(t - t', x - x'; z, z')\big|_{t<t'} = 0.$$

The latter ensures the causal nature of the connection between the perturbation fields and the source and ensures that the appropriate radiation conditions are automatically satisfied. Thanks to the delay condition, the Fourier transformation with respect to time for Green's function can be extended analytically to the upper half-plane of the complex frequencies, which enables its unique expression to be obtained quite easily. For example, in the case of a jump in density of an amount $\gamma \equiv (\rho_2 - \rho_1)/(\rho_2 + \rho_1)$, with $z = 0$ in the remaining uniform unbounded liquid

$$G_{\omega k}(z, z') = \frac{1}{2|k|}\{\gamma\,\mathrm{sgn}\,z\exp[-|k|(|z| + |z'|)] - \exp[-|k|\,|z - z'|]\} + \\ + \frac{1}{2}\gamma g\,\mathrm{sgn}\,z(\gamma + \mathrm{sgn}\,z')\frac{\exp[-|k|(|z| + |z'|)]}{(\omega + i\varepsilon)^2 - \gamma g|k|},$$

$$G(t, x; z, z') = \int\frac{d\omega\,dk}{(2\pi)^2}G_{\omega k}(z, z')\exp(ikx - i\omega t).$$

The velocity potential of a point source of variable amplitude $\varphi = g(x - v_0 t; z, z')\times\exp(-i\omega_0 t)$ moving uniformly at a depth z', can be expressed in terms of the single integral

$$4\pi g(x; z, z') = \ln\{|x^2 + (|z| + |z'|)^2|^{\gamma\,\mathrm{sgn}\,z}\,|x^2 + (z - z')^2|^{-1}\} + \\ + \gamma g\,\mathrm{sgn}\,z(\gamma + \mathrm{sgn}\,z')\int dk\frac{\exp[ikx - |k|(|z| + |z'|)]}{(\omega_0 + kv_0 + i\varepsilon)^2 - \gamma g|k|},$$

the behavior of which is determined by the roots of the denominator as $\varepsilon \to 0$. Their number varies from two to four as a function of the values of the velocity v_0 and the oscillation frequency ω_0. An asymptotic analysis of the solution in the special case of a free surface ($\rho_1 \to 0$) is given in [1-5], and a generalization for internal waves on a jump in density is given in [5-7].

We will now estimate the energy loss per unit time due to radiation of waves from a moving oscillating mass source, averaged over the oscillation period. Using the expression for the pressure in terms of the time derivative of the potential (the linear theory) we obtain an integral representation of the loss as a quadratic form with respect to the source (the calculations are carried out with a real source $\mathrm{Re}\,m = m_0\cos\omega_0 t$)

$$\langle W\rangle \equiv \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} dt\int dx\,dz\,pm = -\int_0^{\infty} d\omega\frac{\omega}{4\pi}\int dz\,dz'\,dk\,\rho(z)\,m_0(-k, z)\,m_0(k, z')\,\mathrm{Im}\,G_{\omega k}(z, z')\,\delta(\omega - |\omega_0 + kv_0|),$$

in which the imaginary part of the Fourier transform of Green's function occurs (it is assumed that $zz' > 0$). It is concentrated on the dispersion surface of the linear interval waves

$$\operatorname{Im} G_{\omega k}(z, z') = -\frac{\pi}{2}\gamma g \operatorname{sgn}(\omega z)(\gamma + \operatorname{sgn} z') \exp[-|k|(|z| + |z'|)]\,\delta(\omega^2 - \gamma g|k|),$$

and finally in the formula for the loss the product of two δ-functions occurs, which ensures that the two integrations over the frequency and the wave number for arbitrary source distributions are satisfied.

Using the known formula of the theory of generalized functions

$$\delta(f(x)) = \sum_i \left|\frac{\partial f}{\partial x}\right|^{-1} \delta(x - x_i), \quad f(x_i) = 0$$

the product of the δ-functions is converted into a sum over four different wave contributions:

$$\delta(\omega - |\omega_0 + kv_0|)\,\delta(\omega^2 - \gamma g|k|) = \sum_{i=1}^{4} \delta(k - k_i)\frac{\delta(\omega - \sqrt{\gamma g|k|})}{2|\omega|\,|v_0 - c_g \operatorname{sgn}(\omega_0 + kv_0)|}, \quad c_g \equiv \frac{\partial \omega}{\partial k},$$

which leads to a corresponding expansion of the loss of energy by a source moving along one of the sides of the layer of the jump in density (for motion upwards $\rho = \rho_1$ and a minus sign and downwards $\rho = \rho_2$ and a plus sign):

$$\langle W \rangle = \langle W_1 + W_2 \rangle H(v_* - v_0) + \langle W_3 + W_4 \rangle,$$

$$\langle W_i \rangle = \frac{\gamma(1 \mp \gamma)\rho g}{16}\frac{|\mu_{0i}|^2}{|v_0 - c_{gi}\operatorname{sgn}(\omega_0 + k_i v_0)|},$$

$$v_* \equiv \frac{\gamma g}{4\omega_0}, \quad \mu_{0i} \equiv \int dx\,dz\,m_0(x, z)\exp(-|k_i z| - ik_i x).$$

For low subcritical velocities of motion of the source ($v_0 < v_*$) the total losses are summed from the contributions of all four systems of waves, corresponding to the two positive and two negative roots:

$$k_1 > k_2 > k_3 > k_4, \quad |\omega_0 + k_i v_0| = \sqrt{\gamma g|k_i|},$$

$$k_{1,2} = \frac{\gamma g}{4v_0^2}\left(1 \pm \sqrt{1 - \frac{v_0}{v_*}}\right)^2, \quad k_{3,4} = -\frac{\gamma g}{4v_0^2}\left(\sqrt{1 + \frac{v_0}{v_*}} \mp 1\right)^2.$$

For one of the two types of waves traveling in the direction of motion (with $k_1 > 0$ and $k_2 > 0$) which are only radiated for subcritical velocities, the group velocity $c_{g2} = c_2/2 = v_0/(1 - \sqrt{1 - v_0/v_*})$, made up of half the phase velocity, exceeds the velocity of the source (the wave-forerunner).

At supercritical velocities ($v_0 > v_*$) the positive roots disappear (become complex) and only radiate waves with negative phase velocities in a direction opposite to the direction of motion of the source. Under stable stratification conditions ($0 < \gamma < 1$) the supercritical mode is more easily achieved for internal waves than for waves on a free surface ($\gamma = 1$).

For all modes on the surface of discontinuity of the density in an unbounded liquid the group velocity c_g is half the phase velocity, and the following relations hold:

$$|v_0 - c_{gi}\operatorname{sgn}(\omega_0 + k_i v_0)| = \frac{1}{2}\left|v_0 - \frac{\omega_0}{k_i}\right|,$$

$$v_0 - \frac{\omega_0}{k_{1,2}} = \pm\sqrt{\frac{\gamma g}{k_{1,2}}\left(1 - \frac{v_0}{v_*}\right)}, \quad v_0 - \frac{\omega_0}{k_{3,4}} = \sqrt{-\frac{\gamma g}{k_{3,4}}\left(1 + \frac{v_0}{v_*}\right)},$$

$$\langle W_i \rangle = \frac{\gamma(1 \mp \gamma)\rho g}{8}\frac{|\mu_{0i}|^2|k_i|}{|v_0 k_i - \omega_0|},$$

which enables us to simplify the comparison of the contributions of different modes to the losses

$$\frac{\langle W_1 \rangle}{\langle W_2 \rangle} = \frac{|\mu_{01}|^2}{|\mu_{02}|^2}\sqrt{\frac{k_1}{k_2}}, \quad \frac{\langle W_3 \rangle}{\langle W_4 \rangle} = \frac{|\mu_{03}|^2}{|\mu_{04}|^2}\sqrt{\frac{k_3}{k_4}},$$

$$\frac{\langle W_1 \rangle}{\langle W_4 \rangle} = \frac{|\mu_{01}|^2}{|\mu_{04}|^2}\sqrt{\frac{k_1}{|k_4|}\frac{v_* + v_0}{v_* - v_0}}.$$

For the simplest multipole sources the ratios $|\mu_{0i}|^2/|\mu_{0,i+1}|^2$ are proportional to the product of power and exponential functions of the wave number. For example, for a point source and a point dipole

$$m_0(x,z) = m_0\delta(x)\delta(z-z_0), \qquad \mu_0 = m_0 \exp(-|kz_0|),$$
$$m_0(x,z) = -d_0\delta'(x)\delta(z-z_0), \qquad \mu_0 = -ikd_0 \exp(-|kz_0|).$$

We will first discuss the simplification for the case of motion close to the surface of the jump ($z_0 \to 0$). For very low velocities ($v_0 \ll v_*$) the two wave vectors $k_1 \approx -k_4 \approx \gamma g/v_0^2 \gg k_2 \approx -k_3$ that are close in value will not be small; they determine the loss in energy

$$\langle W\rangle \approx \frac{(1\mp\gamma)\rho}{4}|\mu_{01}|^2\sqrt{\frac{k_1}{\gamma g}}.$$

In the subcritical band the root k_3 remains much smaller than k_4 ($k_4/k_3 > 5$ even for $v_0 = v_*$). The positive roots merge on approaching the critical velocity ($k_1 \approx k_2 \approx \gamma g/(4v_0^2)$), and their contributions to the losses have a resonance form due to the fact that $v_0 - \omega_0/k_{1,2}$ vanishes. For this reason they are more important than the root of higher value k_4 (when $v_0 \approx v_*$ we have $|k_4| > 5k_1$) and determine the losses

$$\langle W\rangle = \frac{(1\mp\gamma)\gamma g\rho|\mu_{01}|^2}{8v_0}\sqrt{\frac{v_*}{v_*-v_0}}.$$

The singularity requires a more refined nonlinear consideration.

On changing to supercritical velocities the two remaining negative roots ($|k_4| > |k_3|$) merge as the velocity increases and when $v_0 \gg v_*$ they give

$$\langle W\rangle \approx \frac{(1\mp\gamma)\gamma g\rho|\mu_{04}|^2}{8v_0}, \qquad k_3 \approx k_4 \approx -\frac{\omega_0}{v_0}.$$

The change in the relative contributions of different waves with depth of the source is determined by the above-mentioned competition between the power and exponential relationships. For example, for a point dipole

$$\frac{\langle W_3\rangle}{\langle W_4\rangle} = \left|\frac{k_3}{k_4}\right|^{5/2}\exp\{2|z_0|(|k_4|-|k_3|)\}$$

and in the region of the surface a lower contribution will correspond to the longer wave (with lower $|k_3|$). However, because of the slower attenuation with depth of the longer wave it turns out to be more important when the source is fairly deep. Hence, when an oscillating source moves at a depth the main energy losses may be due to the excitation of longer waves.

2. Layers of Finite Depth. There is a quantitative complication if the finite depth of the liquid is taken into account, but the qualitative features remain the same. As before, a moving oscillating source may generate from two to four modes (when the free surface and jump layer is taken into account the number of possibilities is doubled).

We will confine ourselves to the example of a layer of uniform liquid of depth h with a free surface. The average losses of energy, as before, are determined by the imaginary part of the Fourier transform of Green's function

$$\operatorname{Im} G_{\omega k}(z,z') = -\pi g \operatorname{sgn}\omega \frac{\operatorname{ch}|k|(h-z)\operatorname{ch}|k|(h-z')}{\operatorname{ch}^2|k|h}\delta(\omega^2 - g|k|\operatorname{th}kh).$$

Thanks to the two δ-functions the loss integral

$$\langle W\rangle = \frac{g}{4}\int_0^\infty d\omega\int dk\frac{\omega|M_0|^2}{\operatorname{ch}^2|k|h}\delta(\omega^2 - g|k|\operatorname{th}|k|h)\,\delta(\omega - |\omega_0 + kv_0|)$$
$$\left(M_0 \equiv \int_0^h dz m_0(k,z)\operatorname{ch}|k|(h-z)\right)$$

reduces to the sum of the contributions of waves defined as the solutions of the equations

$$\omega = \sqrt{g|k|\operatorname{th}|k|h} = |\omega_0 + kv_0|.$$

Here two or four solutions are also possible depending on the velocity of the source and the pair of positive solutions disappear in the supercritical mode. One of these corresponds to wave-forerunners with a group velocity exceeding the velocity of the source. However, there

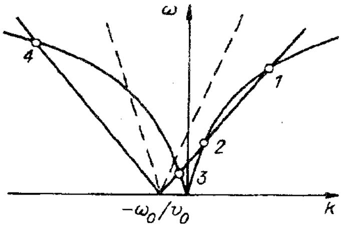

Fig. 1

are no simple explicit analytic formulas for each separate solution, and a graphical representation becomes more convenient.

According to Fig. 1, the solutions $k_1 > k_2 > k_3 > k_4$ are determined by the points of intersection 1-4 of the $\omega = \sqrt{g|k| \tanh |k|h}$ curve with the straight lines $\omega = \pm(\omega_0 + kv_0)$. These straight lines in the supercritical situation, when the points 1 and 2 disappear, are represented by the dashed lines. In parametric form (the dimensionless parameter q is equal to the product of the wave number and the depth) the dependence of the critical velocity v_* on h and ω_0 is represented as follows:

$$\frac{2v_*}{\sqrt{gh}} = \sqrt{\frac{\operatorname{th} q}{q}}\left(1 + \frac{2q}{\operatorname{sh} 2q}\right), \quad 2\omega_0\sqrt{\frac{h}{g}} = q\sqrt{\frac{\operatorname{th} q}{q}}\left(1 - \frac{2q}{\operatorname{sh} 2q}\right).$$

It takes a simple form at large depths and low frequencies:

$$\lim_{h\to\infty} v_* = \frac{g}{4\omega_0}, \quad \lim_{\omega_0\to 0} v_* = \sqrt{gh}.$$

The last relation reflects the well-known fact that in the plane problem when $v_0 > \sqrt{gh}$ there is no radiation of waves by a uniformly moving source of constant amplitude.

The energy losses are the sum of four contributions in the subcritical situation, two of which disappear in the supercritical situation

$$\langle W\rangle = \frac{g}{8}\sum_{i=1}^{4}\frac{|M_{0i}|^2}{|v_0 - c_{gi}\operatorname{sgn}(\omega_0 + k_i v_0)|\operatorname{ch}^2|k_i h|}.$$

Here the denominator can also be converted to the form

$$|v_0 - c_{gi}\operatorname{sgn}(\omega_0 + k_i v_0)| = \frac{|\omega_0^2 - k_i^2 v_0^2 + gh\,k_i^2\operatorname{ch}^{-2}|k_i h||}{2|k_i|\,|\omega_0 + k_i v_0|}.$$

3. Internal Waves in a Waveguide with Solid Covers. In a stratified liquid between horizontal planes with a distribution of the buoyancy frequency N(z) the reaction to a small external action is described by Green's function with the Fourier transformation

$$G_{\omega k}(z, z') = \sum_n \frac{\omega_n^2 \psi_n(|k|, z)\,\psi_n(|k|, z')}{(\omega + i\varepsilon)^2 - \omega_n^2},$$

$$\operatorname{Im} G_{\omega k}(z, z') = -\pi\omega^2 \operatorname{sgn}\omega \sum_n \psi_n(|k|, z)\,\psi_n(|k|, z')\,\delta(\omega^2 - \omega_n^2),$$

where the summation is carried out over wave modes with eigenvalues $\omega_n = \omega_n(|k|)$ and eigenfunctions $\psi_n(|k|, z)$, which satisfy the equation, boundary conditions, and normalization condition

$$(\partial^2/\partial z^2 - k^2 + k^2 N^2(z)/\omega_n^2)\,\psi_n(|k|, z) = 0,$$

$$\psi_n(|k|, h_1) = \psi_n(|k|, h_2) = 0, \quad \int_{h_1}^{h_2} dz\, k^2 N^2(z)\,\psi_n^2(|k|, z) = 1.$$

Perturbations of the vertical component of the velocity w and the pressure p can be expressed as follows in terms of the moving oscillating mass source causing them:

$$p(\omega, k, z) = -\frac{i\omega}{k^2}\left(m(\omega, k, z) - \frac{\partial}{\partial z} w(\omega, k, z)\right),$$

$$w(\omega, k, z) = \omega^2 \int dz'\, m(\omega, k, z')\,\frac{\partial}{\partial z'} G_{\omega k}(z, z'),$$

$$m(t, x, z) = m_0(x - vt, z)\cos\omega_0 t,$$
$$m(\omega, k, z) = \pi m_0(k, z)\,[\delta(\omega - \omega_0 - kv_0) + \delta(\omega + \omega_0 - kv_0)].$$

The average energy losses per unit time as before depend only on the imaginary part of the Fourier transform of Green's function, concentrated on the dispersion surface

$$\langle W\rangle = \sum_n \langle W_n\rangle,$$
$$\langle W_n\rangle = \frac{1}{8}\int dk \int_0^\infty d\omega \frac{\omega^4}{k^2}|M_{0n}|^2\,\delta(\omega - \omega_n(k))\,\delta(\omega - |\omega_0 + kv_0|),$$
$$M_{0n} \equiv \int dz\, m_0(k, z)\frac{\partial\psi_n(|k|, z)}{\partial z} = \int dx\, dz\, m_0(x, z)\frac{\partial\psi_n}{\partial z}\exp(-ikx).$$

The total losses are made up of the losses for the individual modes. Waves with wave vectors which satisfy the equations

$$\omega = \omega_n(k) > 0, \quad \pm\omega = \omega_0 + kv_0,$$

make a nonzero contribution to the energy loss for the n-th wave mode.

For certain simplest types of stratification, to which the uniform stratification (N = const) belongs, the dispersion curves of all the modes $\omega_n = \omega_n(k)$ are convex with $\partial^2\omega_n/\partial k^2 < 0$ and, consequently, can have two points of intersection with the straight line $\omega = \omega_0 + kv_0$. Then, the solution of this system of equations for a fixed number n will be similar to those considered for discontinuous stratification. A graphical representation with a small change due to the frequency limitation $\omega_n < N_{max}$ is, as before, more convenient. For each n-th mode four types of solution are possible, two of which ($k_{1n} > k_{2n} > 0$) disappear for supercritical velocities $v_0 > v_{*n}$, while the contribution to the mean losses due to radiation of waves of this mode can be represented in the form

$$\langle W_n\rangle = \frac{1}{8}\sum_{i=1}^{4}\frac{\omega_{ni}^4|M_{0ni}|^2}{k_{ni}^2|v_0 - c_{gni}\,\mathrm{sgn}(\omega_0 + k_{ni}v_0)|},$$

i.e., as the sum of four (or two in the supercritical case) waves of this mode with wave numbers k_{ni}.

The set of critical velocities v_{*n} is determined by the solution of the following systems of equations:

$$\omega_n(k) = \omega_0 + kv_{*n}, \quad c_{gn} \equiv \frac{\partial\omega_n(k)}{\partial k} = v_{*n}.$$

All the critical velocities are finite. The order of the dispersion curves as the mode number changes implies the order of the critical velocities $v_{*n} > v_{*n+1}$. Finally, as the velocity of motion of the oscillating source decreases, it is possible for an even greater number of wave-forerunners to appear (with $c_{gn}|_{k=k_n} > v_0$).

However, these far from exhaust all possibilities. In many cases the stratification is such that the change in the group velocities c_{gn} with wave number is not monotonic. In addition to the highest maximum for long waves ($k \to 0$) smaller maxima are possible for shorter waves. Their number may increase as the mode number increases. Over a certain range of the parameters v_0, ω_0 an additional pair of wave solutions with positive wave vectors will be associated with each local maximum of the group velocity; one of these wave vectors corresponds to an additional wave-forerunner. Hence, there can also be more than two wave solutions for a fixed mode with positive k. A search for critical velocities both using the equations and graphically is complicated (there may be several of them even for each mode, according to the above discussion). An upper estimate remains simple. The general limitations on the phase and group velocities of the waves in a waveguide (the first is obtained from the comparison theorem [8])

$$\frac{N_{min}^2 h^2}{\pi^2 n^2 + k^2 h^2} \leqslant c_n^2 \leqslant \frac{N_{max}^2 h^2}{\pi^2 n^2 + k^2 h^2},$$
$$\max\left(0,\ 1 - \frac{\omega_n^2}{N_{min}^2}\right) \leqslant \frac{c_{gn}}{c_n} \leqslant 1 - \frac{\omega_n^2}{N_{max}^2}$$

enable us to write

$$v_{*n} < c_{gn}\big|_{k=0} \leqslant \frac{N_{\max} h}{\pi n}.$$

The change in the convexity of the dispersion curves is typical for waveguides with two pronounced maxima of the buoyancy frequency [9]. However, in the case of certain density distributions with a single maximum of the buoyancy frequency, a nonmonotonic dependence of the group velocity on the wave number is possible [10].

4. An Unbounded Uniformly Stratified Liquid. In this example, if there are no limitations on the stratification in the vertical direction, the spectrum of the waves will depend not on the discrete number of modes, but on a continuous parameter — the vertical component of the wave vector. The expression for the average energy loss in the plane problem is not the sum over the modes, but an integral over this component (an analysis of the perturbations in the spatial problem is given in [11]).

In a medium with a constant buoyancy frequency the imaginary part of the Fourier transform of Green's function and the relation between the Fourier transforms of the pressure and the mass source have the form

$$\operatorname{Im} G(\omega, \mathbf{k}) = -\pi \operatorname{sgn} \omega \delta(\omega^2 k^2 - N^2 k_x^2),$$
$$p(\omega, \mathbf{k}) = i\omega (N^2 - \omega^2) G(\omega, \mathbf{k})\, m(\omega, \mathbf{k}).$$

As before, the expression for the average energy losses of a uniformly moving oscillating source can be represented as follows:

$$\langle W \rangle = \int_0^\infty \frac{d\omega}{8\pi} \int d\mathbf{k}\, |\omega|\, (N^2 - \omega^2)\, m_0(\mathbf{k})|^2\, \delta(\omega^2 k^2 - N^2 k_x^2)\, \delta(\omega - |\omega_0 + \mathbf{k}v_i|).$$

Here, of the three integrations, due to the two δ-functions we can carry out two of these and abandon, for example, the integration over the vertical component in the general answer. In the special case of horizontal motion it is determined by the solutions of the following system of equations:

$$\omega = N\frac{|k_x|}{k}, \quad k = |\mathbf{k}|, \quad -\omega_0 \pm \omega = k_x v_0.$$

For a fixed vertical component k_z the solution of this system is exactly the same as it was for discontinuous stratification, and is clearly illustrated in the same figure, changed by taking into account the limitation $\omega < N$. As previously the increase in the number of wave solutions from two to four is determined by the passage of the velocity of motion of the source through the critical value v_*. The function k_z is now the last, i.e., the angle of inclination of the waves to the horizontal. From the system of equations for the critical velocity

$$\omega = N\frac{k_x}{k} = \omega_0 + k_x v_*, \quad \frac{\partial\omega}{\partial k_x} = N\frac{k_z^2}{k^3} = v_*$$

it follows that

$$v_* = \frac{N}{k_z}\left[1 - \left(\frac{\omega_0}{N}\right)^{2/3}\right]^{3/2}.$$

For any velocity of motion of the oscillating source with an oscillation frequency $\omega_0 < N$ we can obtain radiated waves as long as desired, with respect to which the motion will be subcritical ($v_0 < v_*$). Of course, when $\omega_0 < N$ a pair of waves with positive components of k_x and a small value of $|k_z|$ will always be excited, and one of these will be a wave-forerunner.

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PROPAGATION OF A BOUNDARY DISTURBANCE IN A STRATIFIED GAS FOR ARBITRARY KNUDSEN NUMBER

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Introduction. A systematic treatment of wave disturbances in rarefied gases should be based on the Boltzmann kinetic equation or its standard approximations [1, 2]. The purpose of the present paper is to use the kinetic equation to study the forced vibrations of a vertically stratified gas in a gravitational field for given types of excitation on the boundaries. Defining the Knudsen number Kn of the problem as the ratio of the mean free path of the gas molecules to the scale of the inhomogeneities due to the propagating waves, we find that Kn increases with height because of the height dependence of the mean free path in the stratified gas. Hence it is required to determine the motion of the gas for arbitrary Kn.

In many respects this problem is similar to the well-known problem of propagation of ultrasound in a uniform gas. Interest in the latter problem from the point of view of the kinetic theory of gases was stimulated by the work of Van Chan and Uhlenbeck [2]. Important results in this field have been obtained for the linearized Boltzman equation and for approximate kinetic equations using analytic continuation of dispersion relations [3], the Wiener–Hopf method [4], reduction to a Riemann–Hilbert problem [5, 6], and numerical integration along the characteristics [7]. These results suggest that the wave nature of disturbances persists in a gas with $Kn \geq 1$. In this case the phase velocity and absorption coefficient of acoustic waves calculated with the help of the BGK kinetic equation are found to be in good agreement with experiment. The BGK equation can also be used to analyze the propagation of wave disturbances in a stratified gas. Physically, the stratification of the gas leads to internal waves, together with the usual acoustic waves. The dispersion relation for internal waves is quite different from the dispersion relation for acoustic waves and the study of kinetic effects on the propagation of internal waves is of interest in the physics of the upper atmosphere [8]. However, the presence of an external field and the stratification of the gas complicates the problem, since the result is an equation with variable coefficients. Hence the usual methods of finding the solution for sound in a uniform gas are no longer applicable, since they rely on separation of variables with the help of the Fourier transform. The method of integration along the characteristics has to be modified to take into account nonlinear characteristics.

To describe the propagation of boundary disturbances in a stratified gas for arbitrary Knudsen number we reduce the integrodifferential BGK equation to a closed system of integral equations for the first five moments of the distribution function. A general integral kinetic equation including the boundary conditions on the surface of a body in a flowing gas was obtained earlier in [9]. This equation was solved in [10] by transforming to a system of

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